SELF-CONSISTENT MODELS FOR INFINITELY THIN DISCS

Leonid P. Ossipkov

Department of Space Technologies and Applied Astrodynamics,
Saint Petersburg State University, Russia

1 Introduction

It is a fact that the main mass of a typical spiral galaxy is concentrated in a thin disc. A lot of observational data on structure and kinematics of nearby disc galaxies is available now. It will be of great interest to link observations with theories of Galactic Dynamics. The first task is to construct available models of mass distribution. After pioneering works by Wyse & Mayall (1942) and Kuzmin (1944) a theory of mass distribution in self-gravitating discs was developed by many authors (e. g. Fridman & Polyachenko 1976, Binney & Tremaine 1987). The famous Kuzmin (1956) – Toomre (1963) model is an example of such systems. Toomre (1963) discovered a more general family of potential – density pairs. Bisnovatyi-Kogan (1975) and Evans (1993) studied the power-law discs. González & Reina (2006) recently obtained a family of axially symmetric galaxy models with a finite radius. Kutuzov & Ossipkov (1987) using their equipotential method considered discs embedded into a halo, the equipotential sutfaces of the halo being coincided with the equidensities.

To find the steady distribution function (DF) is the final purpose of modelling stellar systems. According to Jeans' theorem as applied to steady stellar discs DF has the form

$$f(r, v_r, v_\theta) = \Psi(E, L).$$

Here r is the distance to the centre of the system, (v_r, v_θ) is the planar velocity vector, E is the energy, L is the modulus of the angular momentum,

$$E = \frac{1}{2}(v_r^2 + v_\theta^2) - \Phi(r), \qquad L = rv_\theta$$

($\Phi(r)$ is the positive potential).

Ng(1967), Lafon (1976), Zweibel (1978) studied disc models starting with a certain expression for $\Psi(E, L)$ (the modified Maxwellian or Schwarzschildian distribution), but the most of authors preferred the way "from the potential to DF". The first models were constructed by Freeman (1966), Bisnovatyi-Kogan & Zel'dovich (1970, 1975), Bisnovatyi-Kogan (1975) and Miyamoto (1971, 1974, 1975). A general theory of equilibrium for collisionless discs was developed by Aoki (1973), Kalnajs (1976), Ossipkov (1978) and Hiotelis & Patsis (1993). Pichon & Lynden-Bell (1996) discussed the kinematics of steady models and constructed DFs for whose Toomre's parameter Q is given.

In notes below we shall consider some DFs for disc systems.

2 General Formulae

We use the following notations:

$\mu(r)$ is the surface density;

$\Phi(r)$ is the potential;

r_* is the boundary radius ($r_* = \infty$ for infinite system);

$V(r)$ is the streaming velocity;

σ_r^2, σ_θ^2 are dispersions of residual velocities.

Then

$$\mu(r) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(r, v_r, v_\theta)\, dv_r\, dv_\theta,$$

$$V(r) = \frac{1}{\mu} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v_\theta f(r, v_r, v_\theta)\, dv_r\, dv_\theta,$$

$$\sigma_r^2 = \frac{1}{\mu} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v_r^2 f(r, v_r, v_\theta)\, dv_r\, dv_\theta,$$

$$\sigma_\theta^2 = \frac{1}{\mu} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (v_\theta - V)^2 f(r, v_r, v_\theta)\, dv_r\, dv_\theta.$$

We shall syppose that $\Phi(r)$ is a monotonously decreasing function. Denote

$$y = 2\left[\Phi(r) - \Phi(r_*)\right].$$

Then y can be used instead of the coordinate r.

Let

$$x = y - v_r^2 - (r/a)^2 v_\theta^2, \qquad \xi = r v_\theta$$

with

$$(r/a)^2 = 1 + \lambda r^2, \qquad \lambda = \text{const} \geq -r_*^{-2}.$$

It is evident that $x = -2E - \lambda\xi^2$ can be considered as the Jacobi integral (for $\lambda > 0$ (Kalnajs 1976). Similar variables were introduced by Veltmann (1965) for spherical systems and were used in the theory of spheres with an ellipsoidal velocity distribution (Ossipkov 1979, Merritt 1985). If $\lambda > 0$, then $\lambda^{-1/2}$ is often called an anisotropy radius.

It is easy to find (e. g. Ossipkov 1978) that

$$\mu < v_r^m v_\theta^n >= (2r^{n+1})^{-1} \iint\limits_{D(r)} \xi^n [x_*(\xi) - x]^{(m-1)/2} \Psi(x, \xi)\, dx\, d\xi$$

where

$$D(r) = \{(x, \xi) : \qquad x \geq 0 \wedge x \leq x_*(\xi)\}$$

and $x_*(\xi) = y - (\xi/a)^2$ is Lindblad's parabola ($v_r = 0$) for the (x, ξ) plane.

Let

$$\Upsilon = \Psi_+(x, \xi) = \frac{1}{2}\left[\Psi(x, \xi) + \Psi(x, -\xi)\right], \quad \Xi = \Psi_-(x, \xi) = \frac{1}{2}\left[\Psi(x, \xi) - \Psi(x, -\xi)\right]$$

be the even and the odd parts of DF. Then

$$\mu(r) = (2r)^{-1} \iint\limits_{D(r)} \Upsilon(x,\xi)\,[x_*(\xi) - x]^{-1/2}\; dx\, d\xi,$$

$$V(r) = (2\mu r^2)^{-1} \iint\limits_{D(r)} \xi\Xi(x,\xi)\,[x_*(\xi) - x]^{-1/2}\; dx\, d\xi,$$

$$\sigma_r^2(r) = (2\mu r)^{-1} \iint\limits_{D(r)} \Upsilon(x,\xi)\,[x_*(\xi) - x]^{1/2}\; dx\, d\xi.$$

One can see that knowledge of the mass distribution $\mu(r)$ or the dispersion of radial velocities σ_r^2 allows to find the even part of DF $\Upsilon(x,\xi)$. Ideas how to find $\Xi(x,\xi)$ were discussed by many authors (e. g. Ossipkov 1978). The problem was studied with details by Kutuzov (1995).

3 Some Simple Models

Now our problem is to invert the integral equation for $\Upsilon(x,\xi)$. It was considered by Kalnajs (1976), Ossipkov (1978), Hiotelis & Patsis (1993), Pichon & Lynden-Bell (1996). The main difficulty lies in a fact the solution is not unique for the equipotential surfaces and the equidensities coincide inside the disc. At first we recall some simple models.

3.1 The ellipsoidal velocity distribution (Kalnajs 1976, Ossipkov 1978)

Let $\Upsilon = \Upsilon(x)$. Denote

$$g(y) = \mu(r)\,(r/a).$$

It is easy to find that

$$g(y) = \frac{\pi}{2}\int\limits_0^y \Upsilon(x)\, dx.$$

Then

$$\Upsilon(x) = \frac{2}{\pi} g'(x).$$

Example 1. . The Maclaurin disc (Ossipkov 1978). Then

$$\Phi = \Phi_0\left[1 - \frac{1}{2}\left(\frac{r}{r_*}\right)^2\right], \quad \mu = \mu_0\left[1 - \left(\frac{r}{r_*}\right)^2\right]^{1/2}, \quad \mu_0 = 2\Phi_0/\pi^2 G r_*.$$

It is easy to find that

$$\Upsilon(x) \propto \left(\frac{x}{\Phi_0}\right)^{-1}\left[(1+\lambda r_*^2) - \lambda r_*^2\left(\frac{x}{\Phi_0}\right)^2\right]^{-1/2}\left[(1+\lambda r_*^2) - 2\lambda r_*^2\left(\frac{x}{\Phi_0}\right)\right].$$

One can see that $\Upsilon \propto x^{-1/2}$ and $\sigma_r^2 = y/3$ (Freeman's model, 1966) for $\lambda = 0$ and $\Upsilon = \text{const}$ and $\sigma_r^2 = y/4$ (the model by Bisnovatyj-Kogan & Zel'dovich 1970) for $\lambda r_*^2 = -1$.

Example 2. The Kuzmin–Toomre disc. Then

$$\Phi = \Phi_0 \left[1 + \left(\frac{r}{r_0}\right)^2\right]^{-1/2}, \quad \mu = \mu_0 \left[1 + \left(\frac{r}{r_0}\right)^2\right]^{-3/2}, \quad \mu_0 = \Phi_0/2\pi G r_0.$$

It is easy to find (Ossipkov 1978) that

$$\Upsilon(x) \propto \left(\frac{x}{\Phi_0}\right) \left[(1 - \lambda r_0^2)\left(\frac{x}{\Phi_0}\right)^2 + 4\lambda r_0^2\right]^{-1/2} \left[3(1 - \lambda r_0^2)\left(\frac{x}{\Phi_0}\right)^2 + 8\lambda r_0^2\right].$$

This expression will be very simple for two cases: $\lambda = 0$ ($\Upsilon \propto x^2$, $\sigma_r^2 = \Phi/4$) and $\lambda r_0^2 = 1$ ($\Upsilon \propto x$, $\sigma_r^2 = \Phi/4$). Note that these models are hot (and Toomre's stability condition is fulfilled).

3.2 The Poincaré (1906) model

Suppose that

$$\Upsilon(x, \xi) = \delta(x)\psi(\xi), \qquad \lambda = -r_*^2.$$

In this model all stars reach the boundary circle. The similar spherical model was qualitively discussed by Poincaré (1906). Then

$$\frac{r}{a}\,\mu = \int\limits_0^{ay^{1/2}} \frac{\psi(\xi)\,d\xi}{(a^2 y - \xi)^{1/2}}$$

Denote

$$t = a^2 y, \qquad \eta = \xi^2, \qquad f(\eta) = \psi(\xi)/2\xi, \qquad h(t) = \frac{r}{a}\,\mu.$$

Thus (Ossipkov 1995)

$$f(\eta) = \frac{1}{\pi}\left[\frac{h(0)}{\eta^{1/2}} + \int\limits_0^{\eta} \frac{h'(t)\,dt}{(\eta - t)^{1/2}}\right].$$

3.3 DF not depending on the energy (Ossipkov 1995)

Now let us consider an opposite case when DF does not depend on our Jacobi integral x (and on the energy in the case $\lambda = 0$). Then

$$\Upsilon(x, \xi) = \psi(x).$$

Denote

$$z = a y^{1/2}, \qquad F(z) = \frac{1}{4}\,\mu(r)\frac{a}{r},$$

and $\psi(\xi)$ can be found from the Abel equation

$$F(z) = \int\limits_0^z \frac{\psi(\xi)\,d\xi}{(\xi - x)^{1/2}}\,.$$

Thus

$$\psi(\xi) = \frac{1}{\pi}\left[\frac{F(0)}{\xi^{1/2}} + \int\limits_0^\xi \frac{F'(z)\,dz}{(\xi - z)^{1/2}}\right].$$

4 Two More General Models

4.1 Dependence on the energy is to be found

Let

$$\Upsilon(x,\xi) = \xi^{2\alpha}\phi(x).$$

Such models were discussed by Ossipkov (1978) and Hiotelis & Patsis (1993). After some simple algebra it can be found that

$$\frac{1}{2}\,\frac{\mu}{r}\,a^{-2\alpha} = \frac{\pi^{1/2}}{2}\,\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\int\limits_0^y (y-x)^\alpha\phi(x)\,dx.$$

Thus the problem is reduced to the generalized Abel equation. Its solution for various α was considered with details by Hiotelis & Patsis (1993).

4.2 Dependence on the angular momentum is to be found

Now

$$\Upsilon(x,\xi) = \psi(\xi)x^\alpha.$$

Denoting

$$z = ay^{1/2}, \qquad F(z) = \frac{1}{2}\,\mu\,\frac{a}{r}\,,$$

we see that

$$F(z) = \frac{\Gamma(\alpha+1)\Gamma(1/2)}{\Gamma(\alpha+3/2)}\int\limits_0^z \frac{\psi(\xi)\,d\xi}{(z^2-\xi^2)^{1/2}}\,,$$

that is the equation of the Abel type.

5 General Considerations

It is not difficult to consider many other DFs for disc models. For instance, the analogue of variables by Kuzmin & Veltmann (1974) can be used. Also, one can image discs with DF of the Gerhard (1990) type.

How to choose DF for real very flattened galaxies? The present day Galactic Dynamics cannot solve this problem. The stability considerations are neccessary but it is not enough. Probably, we must use such observational data as the rotation speed and the run of the line-of-sight velocity dispersion.